CLASSICALLY NORMAL PURE STATES

CHARLES AKEMANN AND NIK WEAVER

Abstract. A pure state f of a von Neumann algebra $\mathcal{M}$ is called *classically normal* if f is normal on any von Neumann subalgebra of $\mathcal{M}$ on which f is multiplicative. Assuming the continuum hypothesis, a separably represented von Neumann algebra M has classically normal, singular pure states iff there is a central projection $p \in M$ such that pMp is a factor of type I_∞, II, or III.

DEFINITION: A pure state f of a von Neumann algebra $\mathcal{M}$ is called *classically normal* if f is normal on any von Neumann subalgebra of $\mathcal{M}$ ("subalgebra" implies the same unit) on which f is multiplicative.

By Lemma 0.2 below, a pure state f on a von Neumann algebra $\mathcal{M}$ is classically normal if, for every von Neumann subalgebra $\mathcal{C}$ of $\mathcal{M}$, either f is not multiplicative on $\mathcal{C}$, or else there is a minimal projection q in $\mathcal{C}$ such that $f(q) = 1$ and q is central in $\mathcal{C}$. Using the continuum hypothesis and a transfinite construction, in Theorem 0.7 we show the existence of classically normal, singular pure states on all infinite dimensional factors acting on a separable Hilbert space. Corollary 0.8 contains the easy "only if" part of the main result.

Here is some history. Let H be a separable infinite-dimensional Hilbert space and let $\mathcal{B}(H)$ denote the algebra of all bounded linear operators on H. Kadison and Singer [12] suggested that every pure state on $\mathcal{B}(H)$ would restrict to a pure state on some maximal abelian self-adjoint subalgebra (aka MASA). Anderson [9] formulated the stronger conjecture that every pure state on $\mathcal{B}(H)$ is of the form $f(a) = \lim_{\mathcal{U}}\langle ae_n, e_n\rangle$ for some orthonormal basis (e_n) and some ultrafilter $\mathcal{U}$ over the natural numbers $\mathbf{N}$. Using the continuum hypothesis, we showed in [6] that these conjectures are false by showing that there is a pure state f on $\mathcal{B}(H)$ that is not multiplicative on any MASA. The argument in the key lemma of that paper used powerful results about the Calkin algebra, so finding the "right" definitions and proofs for general von Neumann algebras took some time.

Our construction of a classically normal pure state will be by transfinite induction, just as in [6]. The difference will be in the proofs of the Lemmas that allow the transfinite construction to go through. We start with some easy facts.

Lemma 0.1. *Let f denote a state on a C^*-algebra $\mathcal{B}$ in which the linear combinations of the projections are dense. f is multiplicative on $\mathcal{B}$ iff $f(p) \in \{0,1\}$ for all projections $p \in \mathcal{M}$.*

Proof. Suppose that $a, b \in B$ and $f(ab) \neq f(ba)$ WLOG we can assume that $a = \sum s_j p_j, b = \sum t_i q_i$, finite linear combinations of projections since the map $(a,b) \to (ab - ba) \to f(ab - ba)$ is continuous, so we only need to show that it annihilates a dense set in $\mathcal{A} \times A$. Then

$$f(ba) = \sum s_j t_j f(q_i p_j).$$

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$$f(ab) = \sum s_j t_j f(p_j q_i).$$

Thus it suffices to prove that for any projections $p, q \in B, f(pq) = f(qp)$. If either $f(p) = 0$ or $f(q) = 0$, then $0 \leq |f(pq)| = |f(qp)| \leq f(p)^{1/2} f(q)^{1/2} = 0$ by the Cauchy-Schwarz inequality. If $f(q) = f(p) = 1$, then $f(1-p) = f(1-q) = 0$, so

$$f(pq) = f(q) - f((1-p)q) = 1 - 0 = 1 = f(p) = f(p) - f((1-q)p) = f(qp).$$

□

Lemma 0.2. *Let f denote a pure state on a von neumann algebra $\mathcal{N}$.*

a. f is normal iff there is a minimal projection $p \in \mathcal{N}$ such that $f(p) = 1$. f is both normal and multiplicative iff the projection p (of the previous sentence) is central.

b. Let $K_f = \{x \in \mathcal{M}$ such that $f(xx^* + x^*x) = 0\}$. Then f is singular iff any increasing, positive approximate unit of K_f converges to 1 in $\mathcal{M}$ for the weak topology.*

Proof. Let f denote a normal pure state and p is its support projection ([19], p. 140) in $\mathcal{N}$. Since the support projection q of f in N^{**} is minimal in N^{**} (see sect. 3.13 of [15]), p must be minimal in $\mathcal{N}$ and $p = q$.

Now suppose that f is a pure state of $\mathcal{N}$ such that there is a minimal projection p in $\mathcal{N}$ such that $f(p) = 1$. Then $f(a) = f(pap)$ by Cauchy-Shwarz inequality. Thus f is normal because $p\mathcal{N}p$ is one dimensional (hence $f|_{p\mathcal{N}p}$ is normal) and $a \to pap$ is weak* continuous.

If p is central, then $f(ab) = f(pappbp) = f(pap)f(pbp) = f(a)f(b)$ because $p\mathcal{N}$ is 1-dimensional.

If f is multiplicative and normal, then $(1-p)\mathcal{N}$ is an ideal and the kernel of f, so $1-p$ is central. This finishes part a.

By part a, if f is not singular (i.e. f is normal), then no approximate unit of K_f could converge weak* to 1 because each element of K_f vanishes on f, and hence $f(1) = 0$, contradicting the assumption that f is a state.

Now suppose that f is singular and some increasing, positive approximate unit (of K_f) $a_\alpha \uparrow r \neq 1$ in the weak* topology of $\mathcal{N}$. By [18], there is a projection $p \leq 1-r$ such that $f(p) = 0$. This contradicts the fact that $\{a_\alpha\}$ is an approximate unit for K_f since $a_\alpha p = 0$ for all α.

□

Lemma 0.3. *A MASA $\mathcal{A}$ of a C^*-algebra $\mathcal{B}$ contains no minimal projections that are not minimal in all of $\mathcal{B}$.*

Proof. If p is a minimal projection of $\mathcal{A}$ and p is not minimal in $\mathcal{B}$, then pBp contains a projection q that is not p or 0. However, for any $a \in A$, $qa - aq = qpa - apq = qpap = papq = q(\lambda p) - (\lambda p)q = 0$ because minimality of p in $\mathcal{A}$ implies that pap is a multiple of p for every $a \in A$.

□

Lemma 0.4. *A pure state f of a von Neumann algebra $\mathcal{M}$ is classically normal if f is normal on any abelian von Neumann subalgebra of $\mathcal{M}$ on which f is multiplicative.*

Proof. This follows immediately from [19] Cor. III.3.11 and the remark following.

□

NOTATION. Now we fix a factor $\mathcal{N}$ of type I_∞, II or III on separable Hilbert space and WLOG assume that a type I_∞ factor is all of $\mathcal{B}(H)$. We let $\mathcal{C}(\mathcal{N})$ denote the set of all von Neumann subalgebras of $\mathcal{N}$.

Lemma 0.5. *Suppose that $\{p_n\}$ is a decreasing sequence of projections in $\mathcal{N}$ that converges to 0 in the weak-operator topology. If $\mathcal{C} \in C(\mathcal{N})$, then there is a singular pure state f on $\mathcal{N}$ and a projection q in $\mathcal{C}$ such that $f(p_n) = 1\ \forall n$ and either*

1. *$f(q) \in (0,1)$, or*
2. *q is minimal in $\mathcal{C}$, q is a central projection in $\mathcal{C}$, and $f(q) = 1$.*

Proof. Working in N^{**} (see [15], sect. 3.8), let $p = \lim\ p_n$. By Corollary 4.5.13 of [15] there exists at least one pure state f of $\mathcal{N}$ such that $f(p) = 1$. Any such f is singular since $p_n \downarrow 0$ in the weak* topology of $\mathcal{N}$. We need to find such an f and a projection $q \in \mathcal{C}$ such that 1 or 2 of the Lemma holds.

Suppose that 1 does not hold for any pure state f of $\mathcal{N}$ with $f(p) = 1$ and projection $q \in \mathcal{C}$. By Lemma 0.1, any pure state f of $\mathcal{N}$ with $f(p) = 1$ is multiplicative on $\mathcal{C}$. (Using [11], Theorem 6.5.2, and a bit more argument, we can assume that $\mathcal{C}$ is abelian, since only the abelian direct summand of a von Neumann algebra can support a nonzero multiplicative linear functional. This is not required for the proof, but it does serve to clarify matters.) We now proceed by cases to reach a contradiction or conclude that condition 2 of the Lemma holds.

Case 1: There is a state g of $\mathcal{C}$ such that, if h is any pure state of $\mathcal{N}$ such that $h(p) = 1$, then $h|_{\mathcal{C}} = g$.

Let Q denote the set of all states e of $\mathcal{N}$ such that $e(p) = 1$. Then Q is convex and weak* compact face of the state space of $\mathcal{N}$ by [5], Theorem 2.10. By the Krein-Milman Theorem, [17],Theorem 3.21, Q is the weak* closed convex hull of its extreme points, and the extreme points of Q are pure states of $\mathcal{N}$. Since any of the extreme points of Q restrict to g on $\mathcal{C}$, the same must be true of all the states of Q. As noted above, g is multiplicative on $\mathcal{C}$.

There are two subcases:

Subcase a. g is normal on $\mathcal{C}$. Choose any pure state f of $\mathcal{N}$ such that $f(p) = 1$. Then q exists satisfying condition 2 of the lemma by Lemma 0.2a.

Let $\mathcal{C}_g = \{a \in \mathcal{C} : g(a) = 0\}$, and let $\{r_\alpha\}$ be an approximate unit in $\mathcal{C}_g$ with

Subcase b. g is singular on $\mathcal{C}$. Let f be a pure state of $\mathcal{N}$ such that $f(p) = 1$. Since $f|\mathcal{C} = g$ is singular, f must also be singular. By part b of Lemma 0.2, if $\{r_\alpha\}$ is an increasing approximate unit for $K_f = \{a \in \mathcal{N} : f(a^*a + aa^*) = 0\}$ with $r_\alpha \uparrow r \in \mathcal{N}^{**}$, then $r_\alpha \to 1$ in the weak* topology of $\mathcal{N}$. Then r is an open projection, hence regular in $\mathcal{N}$ by [1], Prop. II.14 since $\mathcal{N}$ is a von Neumann algebra. Also, if $\overline{r}$ denotes the closure of r in $\mathcal{N}^{**}$, then $\overline{r} \in \mathcal{N}$ by [2], Theorem II.1, so $\overline{r} = 1$.

Thus $\|p_n r p_n\| = \|r p_n\|^2 = 1$ for all n by regularity of r. Consequently by Corollary 4.5.13 of [15], there exist pure states $\{f_n\}$ on $\mathcal{N}$ such that $|f_n(p_n r p_n)| > 1 - 1/n$ and $f_n(p_n) = 1$ for all n. Set $g_n = f_n|_{\mathcal{C}}$. Since any limit point h of $\{f_n\}$ in N^* must satisfy $h(p_n) = 1$ for all n, then $h(p) = 1$. Consequently, by the paragraph following the Case 1 assumption, $h|_C = g$ and $f_n|_C \to g$ in $\mathcal{C}^*$ (weak* topology). However, since $f_n(r) > .5$ for all large n and $g(r) = 0$, we contradict [3], Theorem 4. Thus subcase b leads to a contradiction, so subcase a must hold for Case 1.

Case 2: The remaining possibility is that there are two pure states f, g of $\mathcal{N}$ such that $f|_C \neq g|_{\mathcal{C}}$ and $f(p) = g(p) = 1$. As noted in the paragraph above the Case 1 statement, f and g are multiplicative on $\mathcal{C}$, hence there exists a projection q in $\mathcal{C}$ such

that $f(q) = 1, g(q) = 0$. Since p commutes with $\mathcal{C}$ by Corollary 4.5.13 of [15] and the assumption that property 1 of the Lemma is false, $q - pq = 1 - (\sup(p, (1-q))$, is open by Corolary II.7 of [1]. Since $q - qp_nq \uparrow q - qp$, the operator $b = \sum_1^\infty 2^{-n}(q - qp_nq)$ is strictly positive in the hereditary C*-subalgebra

$$D = \{d \in N : \|(q - qp_nq)d\| + \|d(q - qp_nq)\| \to 0\}.$$

Thus the spectral projections $r_n = q - q\chi_{(1/n, \|b\|]}(b) \downarrow q - (q - qp) = qp$. Similarly we get a sequence $\{s_n\}$ of projections such that $1 - q \geq s_n \downarrow (1 - q)p$. Since $p_n \to 0$ in the weak operator topology, we can pass to a subsequence and assume that $r_n - r_{n+1}$ and $s_n - s_{n+1}$ are all nonzero. To reach a contradiction of the assumption that Case 1 fails, it suffices to find an irreducible representation π of $\mathcal{N}$ such that $\pi^{**}(pq) \neq 0 \neq \pi^{**}(p(1 - q))$.

N.B. Up to this point in the proof, $\mathcal{N}$ could be any von Neumann algebra. We now break the proof into subcases to handle the different types of factors.

Type III subcase: Set $e_n = (r_n - r_{n+1}), e'_n = (s_n - s_{n+1})$. Set $e = \sum_{n=1}^\infty e_n, e' = \sum_{n=1}^\infty e'_n$. Choose partial isometries v_n such that $v_n^* v_n = e'_n, v_n v_n^* = e_n$, and set $v = \sum_1^\infty v_n$ (any two non-trivial projections are equivalent because $\mathcal{N}$ is a type III factor and separably represented). By Corollary 4.5.13 of [15], we can choose an irreducible reepresentation π of $\mathcal{N}$ such that

$$\pi^{**}(pq) = \lim \pi(r_n) \geq \lim_{k\to\infty} \pi(\sum_{n=k}^\infty e_n) \neq 0$$

because $\lim_{k\to\infty} \sum_{n=k}^\infty e_n \neq 0$.

Hence

$$\|\pi^{**}(p(1-q))\| = \|\lim_{k\to\infty} \pi(s_k)\| \geq \|\lim_{k\to\infty} \pi(\sum_{n=k}^\infty e'_n)\| \geq \|\lim_{k\to\infty} \pi(v^*)\pi(\sum_{n=k}^\infty e_n)\pi(v)\|$$

$$= \|\pi(v^*)(\lim_{k\to\infty} \pi(\sum_{n=k}^\infty e_n))\pi(v)\|$$

$$\geq \|\pi(v)\pi(v^*)(\lim_{k\to\infty} \pi(\sum_{n=k}^\infty e_n))\pi(v)\pi(v^*)\| = \|\pi(e)(\lim_{k\to\infty} \pi(\sum_{n=k}^\infty e_n))\pi(e)\|$$

$$= \|(\lim_{k\to\infty} \pi(\sum_{n=k}^\infty e_n))\| \neq 0$$

as was to be shown.

Type II subcase: Let τ denote a normal, semi-finite trace on $\mathcal{N}$. We use the same idea as in the type III case except we use the continuity of the trace to choose $e_n \leq (r_n - r_{n+1}), e'_n \leq (s_n - s_{n+1})$ such that $0 < \tau(e_n) = \tau(e'_n) \leq 2^{-n}$ for all n. Choose partial isometries v_n and define v as in the type III case (because any two projections with the same finite trace are equivalent). The same contradiction then arises as in the type III subcase.

Type I_∞ subcase: Again we mimic the type III case with the following exception. Since $r_n \neq 0$ and $s_n \neq 0$,. We can then choose rank 1 projections $e_n \leq (r_n - r_{n+1}), e'_n \leq (s_n - s_{n+1})$. Choose partial isometries v_n and define v as in the type III case. The contradiction then follows as in the Type III subcase.

□

Lemma 0.6. *Let $\mathcal{A}$ denote a separable C^*-subalgebra of $\mathcal{N}$ with the same unit, and assume that $\mathcal{A}$ contains the compact operators in the type I_∞ case. Let $\mathcal{C} \in C(\mathcal{N})$. Suppose that h is a pure state of $\mathcal{A}$ that annihilates the conpact operators in $\mathcal{A}$. Then there is a singular pure state f on $\mathcal{N}$ that extends h and a projection $q \in \mathcal{C}$ such that either*

1. *$f(q) \in (0,1)$, or*
2. *q is minimal in $\mathcal{C}$, q is a central projection in $\mathcal{C}$, and $f(q) = 1$.*

Proof. Let $\mathcal{A}_h = \{a \in A : h(a^*a+aa^*) = 0\}$. Since $\mathcal{A}_h$ is separable, then it contains a completely positive element a of norm 1.

There are two cases:

Case 1. 0 is an isolated point in the spectrum of a. Applying the functional calculus to a, we can get a projection $r \in K_h$ that acts as a unit for K_h. $1 - r$ must be of infinite rank in $\mathcal{N}$ because by assumption K_h contains all the finite rank projections in $\mathcal{N}$. Let g be any singular pure state of $\mathcal{N}$ such that $g(1-r) = 1$. Since g is singular and $\mathcal{N}$ acts on a separable Hilbert space, by [18] there is a decreasing sequence $\{p_n\}$ of projections in $\mathcal{N}$ such that $p_1 \leq 1-r$ and $p_n \downarrow 0$ in the weak* toplogy of $\mathcal{N}$ and $g(p_n) = 1$ for all n. Let B be the C*-algebra generated by A and $\{p_n\}$. Clearly $h = (1-r)h(1-r)$ and $g = (1-r)g(1-r)$, and on $(1-r)B(1-r) \cap \mathcal{A} = \{\lambda(1-r) : \lambda \in \mathrm{C}\}, h = g$. Thus g extends h. Further, any pure state f of $\mathcal{N}$ such that $f(p_n) = 1$ for all n will extend g (and hence extend h also). Lemma 0.5 now gives the desired f and $q \in \mathcal{C}$.

Case 2. If 0 is not an isolated point in the spectrum of a, then set $p_n = \chi_{[0,1/n]}(a)$. Let B be the C*-algebra generated by $\mathcal{A}$ and $\{p_n\}$. Let g be any extension of h to a pure state of B. Then $g(1-p_n) \leq g(na) = h(na) = nh(a) = 0$, so $g(p_n) = 1$ for all n. We need only show that any pure state f of $\mathcal{N}$ such that $f(p_n) = 1$ for all n will extend g; then Lemma 0.5 gives the desired f and $q \in \mathcal{C}$.

Let f be any pure state of $\mathcal{N}$ such that $f(p_n) = 1$ for all n. Let $c \in \mathcal{A}$. Let $p = \lim p_n$ in $\mathcal{B}^{**} \subset \mathcal{N}^{**}$. Then $\{1-p_n\}$ is an approximate unit for $\{d \in \mathcal{B} : g(dd^* + d^*d) = 0$, so by [4] Proposition 2.2, p is a minimal projection in $\mathcal{B}^{**}$ and $g = pgp$. But $f(p) = 1$ also, so $f|_{\mathcal{B}} = g|_{\mathcal{B}}$, and the lemma follows.

□

Theorem 0.7. *Assume the continuum hypothesis. There is a classically normal, singular pure state f on $\mathcal{N}$. In particular, f is not multiplicative on any MASA of $\mathcal{N}$*

Proof. Let (a_α), $\alpha < \aleph_1$, enumerate the elements of $\mathcal{N}$. Since every von Neumann subalgebra of $\mathcal{N}$ is countably generated, a simple cardinality argument shows that the cardinality of $\mathcal{C}(\mathcal{N})$ is $\aleph_1$. Let $(\mathcal{M}_\alpha)$, $\alpha < \aleph_1$, enumerate $\mathcal{C}(\mathcal{N})$.

We now inductively construct a nested transfinite sequence of unital separable C*-subalgebras $\mathcal{A}_\alpha$ of $\mathcal{N}$ together with pure states f_α on $\mathcal{A}_\alpha$ such that for all $\alpha < \aleph_1$

(1) $a_\alpha \in \mathcal{A}_{\alpha+1}$
(2) if $\beta < \alpha$ then f_α restricted to $\mathcal{A}_\beta$ equals f_β
(3) $\mathcal{A}_{\alpha+1}$ contains a projection $q_\alpha \in \mathcal{M}_\alpha$ such that either $0 < f_{\alpha+1}(q_\alpha) < 1$ or else q_α is minimal and central in $\mathcal{A}_\alpha$ with $f_{\alpha+1}(q_\alpha) = 1$.

Begin by letting $\mathcal{A}_0$ be any separable C*-subalgebra of $\mathcal{N}$ that is unital (and contains $\mathcal{K}(H)$ when $\mathcal{N}$ is type I_∞) and let f_0 be any pure state on $\mathcal{A}_0$ (that annihilates $\mathcal{K}(H)$ when $\mathcal{N}$ is type I_∞). At successor stages, use the last lemma to find a projection $q_\alpha \in \mathcal{M}_\alpha$ and a pure state g on $\mathcal{N}$ such that $g|_{\mathcal{A}_\alpha} = f_\alpha$ and either $0 < g(q_\alpha) < 1$

or else q_α is minimal and central in $\mathcal{A}_\alpha$ and $g(q_\alpha) = 1$. By [8], Lemma 4, there is a separable C*-algebra $\mathcal{A}_{\alpha+1} \subseteq \mathcal{N}$ which contains $\mathcal{A}_\alpha$, a_α, and q_α and such that the restriction $f_{\alpha+1}$ of g to $\mathcal{A}_{\alpha+1}$ is pure. To see this, write $\mathcal{N}$ as the union of a continuous nested transfinite sequence of separable C*-algebras $\mathcal{B}_\gamma$ such that $\mathcal{B}_0$ is the C*-algebra generated by $\mathcal{A}_\alpha$, a_α, and q_α. The cited lemma guarantees that the restriction of g to some $\mathcal{B}_\gamma$ will be pure. Thus the construction may proceed. At limit ordinals α, let $\mathcal{A}_\alpha$ be the closure of $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$. The state f_α is determined by the condition $f_\alpha|_{\mathcal{A}_\beta} = f_\beta$, and it is easy to see that f_α must be pure. (If g_1 and g_2 are states on $\mathcal{A}_\alpha$ such that $f_\alpha = (g_1 + g_2)/2$, then for all $\beta < \alpha$ purity of f_β implies that g_1 and g_2 agree when restricted to $\mathcal{A}_\beta$; thus $g_1 = g_2$.) This completes the description of the construction.

Now define a state f on $\mathcal{N}$ by letting $f|_{\mathcal{A}_\alpha} = f_\alpha$. By the reasoning used immediately above, f is pure, so f is a classically normal, singular pure state.

If $\mathcal{A}$ is a MASA of $\mathcal{N}$, then in the type II or II cases, there are no minimal projections in $\mathcal{N}$, hence none in $\mathcal{A}$, so f can't be multiplicative on $\mathcal{A}$. In the type I case, the only minimal projections are the rank one projections. Since f is singular, it must vanish on rank one projections, hence it can't be multiplicative (and hence normal because it is classically normal) on $\mathcal{A}$. □

Corollary 0.8. *Assuming the continuum hypothesis, a separably represented von Neumann algebra M has classically normal, singular pure states iff there is a central projection $p \in M$ such that pMp is a factor of type I_∞, II, or III.*

Proof. The implication $\leftarrow$ was essentially proved in the last Theorem since any pure state on pMp will have unique pure extension to M.

For the other direction, suppose that no such projection p exists, so that the center of M is infinite dimensional. Suppose that f is a classically normal singular pure state of $\mathcal{N}$. Since any pure state of a C*-algebra is always multiplicative on the center of the algebra, f must be normal on the center of $\mathcal{N}$ by the definition of classically normal. Thus by Lemma 0.2a there is a minimal projection p in the center of $\mathcal{N}$ such that $f(p) = 1$. Since p is minimal in the center, $p\mathcal{N}p$ is a factor. By the assumption of the Corollary, $p\mathcal{N}p$ must be a factor of type I_n for $n < \infty$, i.e. p must be a finite rank projection in $\mathcal{N}$. However, a singular state of $\mathcal{N}$ must vanish on such projections by Takesaki's singularity cirterion [18]. Since p is a finite sum of minimal projections in $\mathcal{N}$ on which f must vanish, $f(p) = 0$. This contradiction completes the proof.

□

REMARK: Since there is a choice to make at each of the $\aleph_1$ steps of the proof, assuming CH, the methods of the last theorem will produce $2^{\aleph_1}$ classically normal pure states on $\mathcal{N}$. Since $\mathcal{N}$ has cardinality $\aleph_1$, the totality of states of $\mathcal{N}$ must be of cardinality $2^{\aleph_1}$. Since any MASA of an infinite factor has (under CH) $2^{\aleph_1}$ distinct singular pure states, each of which has an extension to a pure state of $\mathcal{N}$, there must be $2^{\aleph_1}$ pure states that are not classically normal.

Corollary 0.9. *Let f be a classically normal, singular pure state of $\mathcal{N}$. Let $\mathcal{A} = \{a \in N : f(a^*a + aa^*) = 0\}$. Then A does not have an abelian approximate unit.*

Proof. Suppose the Corollary is false. I.e. f is a classically normal, singular pure state of $\mathcal{N}$ such that $\mathcal{A} = \{a \in N : f(a^*a + aa^*) = 0$ does have an abelian

approximate unit $\{a_\alpha\}$. Let $\mathcal{B}$ be a MASA of $\mathcal{N}$ that contains $\{a_\alpha\}$. Then $\mathcal{B}$ is an abelian von Neumann subalgebra of $\mathcal{N}$ that contains a decreasing excising net $\{1-a_\alpha\}$ for $f|_B$ by [4], Prop. 2.3, and $f|_\mathcal{B}$ must be pure, hence multiplicative. Since f is classically normal, there is a minimal projection $q \in \mathcal{B}$ such that $f(q) = 1$. By Lemma 0.3, q is minimal in $\mathcal{N}$ also, so f is not singular by Lemma 0.2, a contradiction. $\square$

Not all questions of this general type are resolvable by the methods above. For instance, if for each natural number n, H_n is a Hilbert space of dimension n, and if $M = \sum_1^\infty \oplus B(H_n)$, then M does not have any singular, classically normal pure states. However, our methods don't say whether or not M has a pure state that is not multiplicative on any MASA.

We conclude the paper by mentioning an example that shows that there is more to the existence of classically normal pure states than substantial noncommutativity and nonseparability of the underlying algebra.

NOTATION: F_R is the free group on $\mathrm{card}(R) = \aleph_1$ generators and $\mathcal{C}^*(F_R)$ is the corresponding reduced group C*-algebra.

This example is discussed in [16], Cor. 6.7, where it is shown that $\mathcal{C}^*(F_R)$ is nonseparable, but that every abelian subalgebra is separable. However, unlike the situation described in Corollary 0.9, we show in [7] that, if f is a pure state on $\mathcal{C}^*(F_R)$, then $\mathcal{A} = \{a \in C^*(F_R) : f(a^*a + aa^*) = 0\}$ contains a sequential abelian approximate unit.

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Department of Mathematics, University of California, Santa Barbara, CA 93106

Department of Mathematics, Washington University in Saint Louis, Saint Louis, MO 63130

E-mail address: `akemann@math.ucsb.edu, nweaver@math.wustl.edu`